

Choice and Chance

WIT'S END

The Princess Bride is a brilliant whimsical comedy; among its many memorable scenes, the battle of wits between the hero (Westley) and a villain (the Sicilian Vizzini) ranks high. Westley challenges Vizzini to the following game. Westley will poison one of two glasses of wine out of Vizzini's sight. Then Vizzini will choose to drink from one, and Westley must drink from the other. Vizzini claims to be far smarter than Westley: "Have you ever heard of Plato, Aristotle, Socrates?...Morons." He therefore believes he can win by reasoning.

All I have to do is divine from what I know of you: are you the sort of man who would put the poison into his own goblet or his enemy's? Now, a clever man would put the poison into his own goblet, because he would know that only a great fool would reach for what he was given. I am not a great fool, so I can clearly not choose the wine in front of you. But you must have known I was not a great fool, you would have counted on it, so I can clearly not choose the wine in front of me.

He goes on to other considerations, all of which go in similar logical circles. Finally he distracts Westley, switches the goblets, and laughs confidently as both

drink from their respective glasses. He says to Westley: “You fell victim to one of the classic blunders. The most famous is ‘Never get involved in a land war in Asia,’ but only slightly less well known is this: ‘Never go in against a Sicilian when death is on the line.’” Vizzini is still laughing at his expected victory when he suddenly falls over dead.

Why did Vizzini’s reasoning fail? Each of his arguments was innately self-contradictory. If Vizzini reasons that Westley would poison goblet A, his deduction is that he should choose goblet B. But Westley can also make the same logical deduction, in which case he should poison goblet B. But Vizzini should foresee this, and therefore should choose goblet A. But...There is no end to this circle of logic.*

Vizzini’s dilemma arises in many games. Imagine you are about to shoot a penalty kick in soccer. Do you shoot to the goalie’s left or right? Suppose some consideration—your being left-footed versus right-footed, the goalie being left-handed versus right-handed, or which side you chose the last time you took a penalty kick—suggests that you should choose left. If the goalie is able to think through this thinking, then he will mentally and even physically prepare to cover that side, so you will do better by choosing his right instead. But what if the goalie raises his level of thinking one notch? Then you would have done better by sticking to the initial idea of kicking to his left. And so on. Where does it end?

The only logically valid deduction in such situations is that if you follow any system or pattern in your choices, it will be exploited by the other player to his advantage and to your disadvantage; therefore you should not follow any such system or pattern. If you are known to be a left-side kicker, goalies will cover that side better and save your kicks more often. You have to keep them guessing by being unsystematic, or random, on any single occasion. Deliberately choosing your actions at random may seem irrational in something that purports to be rational strategic thinking, but there is method in this apparent madness. The value of randomization can be quantified, not merely understood in a vague general sense. In this chapter we will explicate this method.

MIXING IT UP ON THE SOCCER FIELD

The penalty kick in soccer is indeed the simplest and the best-known example of the general situation requiring random moves or, in gametheoretic jargon, *mixed strategies*. It has been much studied in theoretical and empirical research on games and discussed in the media.¹

A penalty is awarded for a specified set of prohibited actions or fouls by the defense in a marked rectangular area in front of its goal. Penalty kicks are also used as a tiebreaker of last resort at the end of a soccer match. The goal is 8 yards wide and 8 feet high. The ball is put on a spot 12 yards from the goal line directly in front of the midpoint of the goal. The kicker has to shoot the ball directly from this spot. The goalie has to stand on the goal line at the midpoint of the goal and is not allowed to leave the goal line until the kicker strikes the ball.

A well-kicked ball takes only two-tenths of a second to go from the spot to the goal line. A goalie who waits to see which way the ball has been kicked cannot hope to stop it unless it happens to be aimed directly at him. The goal area is wide; therefore the goalie must decide in advance whether to jump to cover one side and, if so, whether to jump left or right. The kicker in his run up to the spot must also decide which way to kick before he sees which way the goalie is leaning. Of course each will do his best to disguise his choice from the other. Therefore the game is best regarded as one with simultaneous moves. In fact, it is rare for the goalie to stand in the center without jumping left or right, and also relatively rare for the kicker to kick to the center of the goal, and such behavior can also be explained theoretically. Therefore we will simplify the exposition by limiting each player to just two choices. Since kickers usually kick using the inside of their foot, the natural direction of kicking for a right-footed kicker is to the goalie's right, and for a left-footed kicker it is to the goalie's left. For simplicity of writing we will refer to the natural side as "Right." So the choices are Left and Right for each player. When the goalie chooses Right, it means the kicker's natural side.

With two choices for each player and simultaneous moves, we can depict the outcomes in the usual 2-by-2 game payoff table. For each combination of choices of Left and Right by each of the two players, there is still some element of chance; for example, the kick may sail over the crossbar, or the goalie may touch the ball only to deflect it into the net. We measure the kicker's payoff by the percentage of times a goal is scored for that combination of choices, and the goalie's payoff by the percentage of times a goal is not scored.

Of course these numbers are specific to the particular kicker and the goalie, and detailed data are available from the top professional soccer leagues in many countries. For illustrative purposes, consider the average over a number of different kickers and goalies, collected by Ignacio Palacios-Huerta, from the top Italian, Spanish, and English leagues for the period 1995–2000. Remember that in each cell, the payoff shown in the southwest corner belongs to the row player (kicker), and that shown in the northeast corner belongs to the column player (goalie). The kicker's payoffs are higher when the two choose the opposite sides

than when they choose the same side. When the two choose opposite sides, the kicker's success rate is almost the same whether the side is natural or not; the only reason for failure is a shot that goes too wide or too high. Within the pair of outcomes when the two choose the same side, the kicker's payoff is higher when he chooses his natural side than when he chooses his non-natural side. All of this is quite intuitive.

		Goalie	
		Left	Right
Kicker	Left	58 42	95 5
	Right	93 7	70 30

Let us look for a Nash equilibrium of this game. Both playing Left is not an equilibrium because when the goalie is playing Left, the kicker can improve his payoff from 58 to 93 by switching to Right. But that cannot be an equilibrium either, because then the goalie can improve his payoff from 7 to 30 by switching to Right also. But in that case the kicker does better by switching to Left, and then the goalie does better by also switching to Left. In other words, the game as depicted does not have a Nash equilibrium at all.

The cycles of switching neatly follow the cycles of Vizzini's circular logic as to which goblet would contain the poison. And the fact that the game has no Nash equilibrium in the stated pairs of strategies is exactly the gametheoretic statement of the importance of mixing one's moves. What we need to do is to introduce mixing as a new kind of strategy and then look for a Nash equilibrium in this expanded strategy set. To prepare for that, we will refer to the strategies originally specified—Left and Right for each player—as the *pure strategies*.

Before we proceed with the analysis, let us simplify the game table. This game has the special feature that the two players' interests are exactly opposed. In each cell, the goalie's payoff is always 100 minus the kicker's payoff. Therefore, comparing cells, whenever the kicker has a higher payoff the goalie has a lower payoff, and vice versa.

Many people's raw intuition about games, derived from their experience of sports just like this one, is that each game must have a winner and a loser. However, in the general world of games of strategy, such games of pure conflict are relatively rare. Games in economics, where the players engage in voluntary trade for mutual benefit, can have outcomes where everyone wins. Prisoners'

dilemmas illustrate situations where everyone can lose. And bargaining and chicken games can have lopsided outcomes in which one side wins at the expense of the other. So most games involve a mixture of conflict and common interest. However, the case of pure conflict was the first to be studied theoretically and retains some special interest. As we have seen, such games are called zero-sum, the idea being that the payoff of one player is always exactly the negative of that of the other player, or, more generally, *constant-sum*, as in the present case, where the two players' payoffs always sum to 100.

The game table for such games can be simplified in appearance by showing only one player's payoff, since the other's payoff may be understood as the negative of that of the first player or as a constant (such as 100) minus the first player's payoff, as is the case in this example. Usually the row player's payoff is shown explicitly. In that case, the row player prefers outcomes with the larger numbers, and the column player prefers outcomes with the smaller numbers. With this convention, the payoff table for the penalty kick game looks like this:

		Goalie	
		Left	Right
Kicker	Left	58	95
	Right	93	70

If you are the kicker, which of the two pure strategies would you prefer? If you choose your Left, the goalie can keep your success percentage down to 58 by choosing his Left; if you choose your Right, the goalie can keep your success percentage down to 70 by choosing his Right also.* Of the two, you prefer the (Right, Right) combination.

Can you do better? Suppose you choose Left or Right at random in proportions of 50:50. For example, as you stand ready to run up and kick, you toss a coin in the palm of your hand out of the goalie's sight and choose Left if the coin shows tails and Right if it shows heads. If the goalie chooses his Left, your mixture will succeed $1/2 \times 58 + 1/2 \times 93 = 75.5$ percent of the time; if the goalie chooses his Right, your mixture will succeed $1/2 \times 95 + 1/2 \times 70 = 82.5$ percent of the time. If the goalie believes you are making your choice according to such a mixture, he will choose his Left to hold your success rate down to 75.5 percent. But that is still better than the 70 you would have achieved by using the better of your two pure strategies.

An easy way to check whether randomness is needed is to ask whether there is any harm in letting the other player find out your actual choice *before* he responds. When this would be disadvantageous to you, there is advantage in randomness that keeps the other guessing.

Is 50:50 the best mixture for you? No. Try a mixture where you choose your Left 40 percent of the time and your Right 60 percent of the time. To do so, you might take a small book in your pocket, and as you stand ready to run up for your kick, take it out and open it at a random page out of the goalie's sight. If the last digit of the page number is between 1 and 4, choose your Left; if it is between 5 and 0, choose your Right. Now the success rate of your mixture against the goalie's Left is $0.4 \times 58 + 0.6 \times 93 = 79$, and against the goalie's Right it is $0.4 \times 95 + 0.6 \times 70 = 80$. The goalie can hold you down to 79 by choosing his Left, but that is better than the 75.5 percent you could have achieved with a 50:50 mix.

Observe how the successively better mixture proportions for the kicker are narrowing the difference between the success rates against the goalie's Left and Right choices: from the 93 to 70 difference for the better of the kicker's two pure strategies, to the 82.5 to 75.5 difference for the 50:50 mix, to the 80 to 79 difference for the 40:60 mix. It should be intuitively clear that your best mixture proportion achieves the same rate of success whether the goalie chooses his Left or his Right. That also fits with the intuition that mixing moves is good because it prevents the other player from exploiting any systematic choice or pattern of choices.

A little calculation, which we postpone to a later section of this chapter, reveals that the best mixture for the kicker is to choose his Left 38.3 percent of the time and his Right 61.7 percent of the time. This achieves a success rate of $0.383 \times 58 + 0.617 \times 93 = 79.6$ percent against the goalie's Left, and $0.383 \times 95 + 0.617 \times 70 = 79.6$ percent against the goalie's Right.

What about the goalie's strategy? If he chooses the pure strategy Left, the kicker can achieve 93 percent success by choosing his own Right; if the goalie chooses his pure strategy Right, the kicker can achieve 58 percent success by choosing his own Left. By mixing, the goalie can hold the kicker down to a much lower success rate. The best mixture for the goalie is one that gives the kicker the same success rate whether he chooses to kick to the Left or the Right. It turns out that the goalie should choose the proportions of his Left and Right at 41.7 and 58.3, respectively, and this gives the kicker a success rate of 79.6 percent.

Notice one seeming coincidence: the success percentage the kicker can ensure by choosing his best mixture, namely 79.6, is the same as the success

percentage to which the goalie can hold down the kicker by choosing his own best mixture. Actually this is no coincidence; it is an important general property of mixed strategy equilibria in games of pure conflict (zero-sum games).

This result, called the minimax theorem, is due to Princeton mathematician and polymath John von Neumann. It was later elaborated by him, in coauthorship with Princeton economist Oscar Morgenstern, in their classic book *Theory of Games and Economic Behavior*,² which can be said to have launched the whole subject of game theory.

The theorem states that in zero-sum games in which the players' interests are strictly opposed (one's gain is the other's loss), one player should attempt to minimize his opponent's maximum payoff while his opponent attempts to maximize his own minimum payoff. When they do so, the surprising conclusion is that the minimum of the maximum (minimax) payoffs equals the maximum of the minimum (maximin) payoffs. The general proof of the minimax theorem is quite complicated, but the result is useful and worth remembering. If all you want to know is the gain of one player or the loss of the other when both play their best mixes, you need only compute the best mix for one of them and determine its result.

Theory and Reality

How close is the performance of actual kickers and goalies to our theoretical calculations about the respective best mixtures? The following table is constructed from Palacios-Huerta's data and our calculations.³

Proportion of Left in mixture for			The percent of times a goal results when the other player chooses his	
			Left	Right
Kicker	Best	38.3 %	79.6 %	79.6 %
	Actual	40.0 %	79.0 %	80.0 %
Goalie	Best	41.7 %	79.6 %	79.6 %
	Actual	42.3 %	79.3 %	79.7 %

Pretty good, huh? In each case, the actual mixture proportions are quite close to the best. The actual mixtures yield almost equal success rates regardless of the other player's choice and therefore are close to being immune to exploitation by the other.

Similar evidence of agreement between actual play and theoretical

predictions comes from top-level professional tennis matches.⁴ This is to be expected. The same people regularly play against one another and study their opponents' methods; any reasonably obvious pattern would be noticed and exploited. And the stakes are large, in terms of money, achievement, and fame; therefore the players have strong incentives not to make mistakes.

However, the success of game theory is not complete or universal. Later in this chapter we will examine how well or poorly the theory of mixed strategies succeeds in other games and why. First let us summarize the general principle expressed here in the form of a rule for action:

RULE 5: In a game of pure conflict (zero-sum game), if it would be disadvantageous for you to let the opponent see your actual choice in advance, then you benefit by choosing at random from your available pure strategies. The proportions in your mix should be such that the opponent cannot exploit your choice by pursuing any particular pure strategy from the ones available to him—that is, you get the same average payoff when he plays any of his pure strategies against your mixture.

When one player follows this rule, the other cannot do any better by using one of his own pure strategies than another. Therefore he is indifferent between them and cannot do any better than to use the mixture prescribed for him by the same rule. When both follow the rule, then neither gets any better payoff by deviating from this behavior. This is just the definition of Nash equilibrium from chapter 4. In other words, what we have when both players use this rule is a Nash equilibrium in mixed strategies. So, the von Neumann-Morgenstern minimax theorem can be regarded as a special case of the more general theory of Nash. The minimax theorem works only for two-player zero-sum games, while the Nash equilibrium concept can be used with any number of players and any mixture of conflict and common interest in the game.

Equilibria of zero-sum games don't necessarily have to involve mixed strategies. As a simple example, suppose the kicker has very low success rates when kicking to the Left (his non-natural side) even when the goalie guesses wrong. This can happen because there is a significant probability that the kicker will miss the target anyway when kicking with the outside of his foot. Specifically, suppose the payoff table is:

		Goalie	
		Left	Right
Kicker	Left	38	65
	Right	93	70

Then the strategy Right is dominant for the kicker, and there is no reason to mix. More generally, there can be equilibria in pure strategies without dominance. But this is no cause for concern; the methods for finding mixed strategy equilibria will also yield such pure strategy equilibria as special cases of mixtures, where the proportion of the one strategy in the mixture is the entire 100 percent.

CHILD'S PLAY

On October 23, 2005, Andrew Bergel of Toronto was crowned that year's Rock Paper Scissors International World Champion and received the Gold Medal of the World RPS Society. Stan Long of Newark, California, won the silver medal, and Stewart Waldman of New York the bronze.

The World RPS Society maintains a web site, www.worldrps.com, where the official rules of play and various guides to strategy are posted. It also holds an annual world championship event. Did you know that the game you played as a child has become this big?

The rules of the game are the same as the ones you followed as a kid and as described in chapter 1. Two players simultaneously choose ("throw," in the technical jargon of the game) one of three hand signals: Rock is a fist, Paper is a horizontal flat palm, and Scissors is signified by holding out the index and middle fingers at an angle to each other and pointing toward the opponent. If the two players make the same choice, it is a tie. If the two make different choices, Rock wins against (breaks) Scissors, Scissors wins against (cuts) Paper, and Paper wins against (covers) Rock. Each pair plays many times in succession, and the winner of a majority of these plays is the winner of that match.

The elaborate rules set out on the web page of the World RPS Society ensure two things. First, they describe in precise terms the hand shapes that constitute each type of throw; this prevents any attempts at cheating, where one player makes some ambiguous gesture, which he later claims to be the one that beats what his opponent chose. Second, they describe a sequence of actions, called the

prime, the approach, and the delivery, intended to ensure that the two players' moves are simultaneous; this prevents one player from seeing in advance what the other has done and making the winning response to it.

Thus we have a two-player simultaneous-move game with three pure strategies for each. If a win counts as 1 point, a loss as -1 , and a tie as 0, the game table is as follows, with the players named Andrew and Stan in honor of their achievements in the 2005 World Championships:

		Stan's choice		
		Rock	Paper	Scissors
Andrew's choice	Rock	0 0	-1 1	1 -1
	Paper	1 -1	0 0	-1 1
	Scissors	-1 1	1 -1	0 0

What would game theory recommend? This is a zero-sum game, and revealing your move in advance can be disadvantageous. If Andrew chooses just one pure move, Stan can always make a winning response and hold Andrew's payoff down to -1 . If Andrew mixes the three moves in equal proportions of $1/3$ each, it gives him the average payoff of $(1/3) \times 1 + (1/3) \times 0 + (1/3) \times (-1) = 0$ against any one of Stan's pure strategies. With the symmetric structure of the game, this is quite obviously the best that Andrew can do, and calculation confirms this intuition. The same argument goes for Stan. Therefore mixing all three strategies in equal proportions is best for both and yields a Nash equilibrium in mixed strategies.

However, this is not how most participants in the championships play. The web site labels this the Chaos Play and advises against it. "Critics of this strategy insist that there is no such thing as a random throw. Human beings will always use some impulse or inclination to choose a throw, and will therefore settle into unconscious but nonetheless predictable patterns. The Chaos School has been dwindling in recent years as tournament statistics show the greater effectiveness of other strategies."

The problem of "settling into unconscious but nonetheless predictable patterns" is indeed a serious one deserving further discussion, to which we will turn in a moment. But first let us see what kinds of strategies are favored by participants in the World RPS Championship.

The web site lists several “gambits,” like the cleverly named strategy Bureaucrat, which consists of three successive throws of Paper, or Scissor Sandwich, which consists of Paper, Scissors, Paper. Another is the Exclusion Strategy, which leaves out one of the throws. The idea behind these is that opponents will focus their entire strategy on predicting when the pattern will change, or when the missing throw will appear, and you can exploit this weakness in their reasoning.

There are also physical skills of deception, and detection of the opponent’s deception. The players watch each other’s body language and hands for signals of what they are about to throw; they also try to deceive the opponent by acting in a way that suggests one throw and choosing a different one instead. Soccer penalty kickers and goalies similarly watch each other’s legs and body movements to guess which way the other will go. Such skills matter; for example, in the penalty shoot-out that decided the 2006 World Cup quarter-final match between England and Portugal, the Portuguese goalie guessed correctly every time and saved three of the kicks, which clinched the victory for his team.

MIXING IT UP IN THE LABORATORY

By contrast with the remarkable agreement between theory and reality of mixed strategies on the soccer field and the tennis court, the evidence from laboratory experiments is mixed or even negative. The first book-length treatment of experimental economics declared flatly: “Subjects in experiments are rarely (if ever) observed flipping coins.”⁵ What explains this difference?

Some of the reasons are the same as those discussed in chapter 4 when we contrasted the two kinds of empirical evidence. The laboratory setting involves somewhat artificially structured games played by novice subjects for relatively small stakes, whereas the field setting has experienced players engaged in familiar games, for stakes that are huge in terms of fame and prestige and often also in terms of money.

Another limitation of the experimental setting may be at work. The experiments always begin with a session where the rules are carefully explained, and the experimenters go to great lengths to ensure that the subjects understand the rules. The rules make no explicit mention of the possibility of randomization and don’t provide coins or dice or the instruction, “You are allowed, if you wish, to flip the coins or roll the dice to decide what you are going to do.” Then it is hardly surprising that the subjects, instructed to follow the rules precisely as stated, don’t flip coins. We have known ever since Stanley Milgram’s renowned

experiment that subjects treat experimenters as authority figures to be obeyed.⁶ It is hardly surprising that they follow the rules literally and do not think of randomizing.

However, the fact remains that even when the laboratory games were structured to be similar to soccer penalty kicks, where the value of mixing moves is evident, the subjects do not seem to have used randomization either correctly or appropriately over time.⁷

Thus we have a mixed record of success and failure for the theory of mixed strategies. Let us develop some of these findings a little further, both to understand what we should expect in games we observe and to learn how to play better.

HOW TO ACT RANDOMLY

Randomization does not mean alternating between the pure strategies. If a pitcher is told to mix fastballs and forkballs in equal proportions, he should not throw a fastball, then a forkball, then a fastball again, and so on in strict rotation. The batters will quickly notice the pattern and exploit it. Similarly, if the proportion of fastballs to forkballs is to be 60:40, that does not mean throwing six fastballs followed by four forkballs and so on.

What should the pitcher do when mixing fastballs and forkballs randomly in equal proportions? One way is to pick a number at random between 1 and 10. If the number is 5 or less, throw a fastball; if the number is 6 or above, go for the forkball. Of course, this only reduces the problem one layer. How do you go about picking a random number between 1 and 10?

Let us start with the simpler problem of trying to write down what a random sequence of coin tosses will look like. If the sequence is truly random, then anyone who tries to guess what you write down will be correct no more than 50 percent on average. But writing down such a “random” sequence is more difficult than you might imagine.

Psychologists have found that people tend to forget that heads is just as likely to be followed by heads as by tails; therefore they have too many reversals, and too few strings of heads, in their successive guesses. If a fair coin toss comes up heads thirty times in a row, the next toss is still equally likely to be heads or tails. There is no such thing as “being due” for tails. Similarly, in the lottery, last week’s number is just as likely to win again as any other number.

The knowledge that people fall into the error of too many reversals explains many of the stratagems and gambits used by participants in the World RPS

Championships. Players attempt to exploit this weakness and, at the next higher level, attempt to exploit these attempts in turn. The player who throws Paper thrice in succession is looking for the opponent to think that a fourth Paper is unlikely, and the player who leaves out one of the throws and mixes among just the other two in many successive plays is trying to exploit the opponent's thinking that the missing throw is "due."

To avoid getting caught putting order into the randomness, you need a more objective or independent mechanism. One such trick is to choose some fixed rule, but one that is both secret and sufficiently complicated that it is difficult to discover. Look, for example, at the length of our sentences. If the sentence has an odd number of words, call it heads; if the sentence length is even, call it tails. That should be a good random number generator. Working backward over the previous ten sentences yields T, H, H, T, H, H, H, H, T, T. If our book isn't handy, don't worry; we carry random number sequences with us all the time. Take a succession of your friends' and relatives' birthdates. For even dates, guess heads; for odd, tails. Or look at the second hand on your watch. Provided your watch is not too accurate, no one else will know the current position of the second hand. Our advice to the pitcher who must mix in proportions of 50:50 or to the catcher who is calling the pitches is to glance at his wristwatch just before each pitch. If the second hand points toward an even number, then throw a fastball; an odd number, then throw a forkball. The second hand can be used to achieve any ratio. To throw fastballs 40 percent of the time and forkballs 60 percent, choose fastball if the second hand is between 1 and 24, and forkball if it is between 25 and 60.

Just how successful were the top professionals in tennis and soccer at correct randomization? The analysis of data in grand slam tennis finals revealed that there was indeed some tendency to reverse between serves to the forehand and the backhand more frequently than appropriate for true randomness; in the jargon of statistics, there was negative serial correlation. But it seems to have been too weak to be successfully picked up and exploited by the opponents, as seen from the statistically insignificant difference of success rates of the two kinds of serves. In the case of soccer penalty kicks, the randomization was close to being correct; the incidence of reversals (negative serial correlation) was statistically insignificant. This is understandable; successive penalty kicks taken by the same player come several weeks apart, so the tendency to reverse is likely to be less pronounced.

The championship-level players of Rock Paper Scissors seem to place a lot of importance on strategies that deliberately depart from randomization, and try to exploit the other player's attempts to interpret patterns. How successful are

these attempts? One kind of evidence would come from consistency of success. If some players are better at deploying nonrandom strategies, they should do well in contest after contest, year after year. The World RPS Society does not “have the manpower to record how each competitor does at the Championships and the sport is not developed enough so that others track the info. In general, there have not been too many consistent players in a statistically significant way, but the Silver medalist from 2003 made it back to the final 8 the following year.”⁸ This suggests that the elaborate strategies do not give any persistent advantage.

Why not rely on the other player’s randomization? If one player is using his best mix, then his success percentage is the same no matter what the other does. Suppose you are the kicker in the soccer example, and the goalie is using his best mix: Left 41.7 percent and Right 58.3 percent of the time. Then you will score a goal 79.6 percent of the time whether you kick to the Left, the Right, or any mixture of the two. Observing this, you might be tempted to spare yourself the calculation of your own best mix, just stick to any one action, and rely on the other player using his best mix. The problem is that unless you use your best mix, the other does not have the incentive to go on using his. If you stick to the Left, for example, the goalie will switch to covering the Left also. The reason why you *should* use your best mix is to *keep* the other player using his.

Unique Situations

All of this reasoning makes sense in games like football, baseball, or tennis, in which the same situation arises many times in one game, and the same players confront each other from one game to the next. Then there is time and opportunity to observe any systematic behavior and respond to it. Correspondingly, it is important to avoid patterns that can be exploited and stick to the best mix. But what about games that are played just once?

Consider the choices of points of attack and defense in a battle. Here the situation is usually unique, and the other side cannot infer any systematic pattern from your previous actions. But a case for random choice arises from the possibility of espionage. If you choose a definite course of action, and the enemy discovers what you are going to do, he will adapt his course of action to your maximum disadvantage. You want to surprise the enemy; the surest way to do so is to surprise yourself. You should keep your options open as long as possible, and at the last moment choose between them by an unpredictable and, therefore, espionage-proof device. The relative proportions of the device should also be

such that if the enemy discovered them, he would not be able to turn the knowledge to his advantage. This is just the best mix calculated in the description above.

Finally, a warning. Even when you are using your best mix, there will be occasions when you have a poor outcome. Even if the kicker is unpredictable, sometimes the goalie will still guess right and save the shot. In football, on third down and a yard to go, a run up the middle is the percentage play; but it is important to throw an occasional bomb to keep the defense honest. When such a pass succeeds, fans and sportscasters will marvel at the cunning choice of play and say the coach is a genius. When it fails, the coach will come in for a lot of criticism: how could he gamble on a long pass instead of going for the percentage play?

The time to justify the coach's strategy is *before* using it on any particular occasion. The coach should publicize the fact that mixing is vital; that the run up the middle remains such a good percentage play precisely because some defensive resources must be diverted to guard against the occasional costly bomb. However, we suspect that even if the coach shouts this message in every newspaper and on every TV channel before the game, and then uses a bomb in such a situation and it fails, he will come in for just as much criticism as if he had not tried to educate the public in the elements of game theory.

MIXING STRATEGIES IN MIXED-MOTIVES GAMES

In this chapter thus far, we have considered only games where the players' motives are in pure conflict, that is, zero-sum or constant-sum games. But we have always emphasized that most games in reality have aspects of common interests as well as conflict. Does mixing have a role in these more general non-zero-sum games? Yes, but with qualifications.

To illustrate this, let us consider the hunting version of the battle of the sexes game from chapter 4. Remember our intrepid hunters Fred and Barney, who are deciding separately, each in his own cave, whether to go stag hunting or bison hunting that day. A successful hunt requires effort from both, so if the two make opposite choices, neither gets any meat. They have a common interest in avoiding such outcomes. But between the two successful possibilities where they are in the same hunting ground, Fred prefers stag meat and rates the outcome of a jointly conducted stag hunt 4 instead of 3, while Barney has the opposite preferences. Therefore the game table is as shown below.

		Barney's choice	
		Stag	Bison
Fred's choice	Stag	4, 3	0, 0
	Bison	0, 0	3, 4

We saw that the game has two Nash equilibria, shown shaded. We would now call these equilibria in pure strategies. Can there be equilibria with mixing?

Why would Fred choose a mixture? Perhaps he is uncertain about Barney's choice. If Fred's subjective uncertainty is such that he thinks the probabilities of Barney choosing Stag and Bison are y and $(1-y)$, respectively, then he expects the payoff of $4y + 0(1-y) = 4y$ if he himself chooses Stag, and $0y + 3(1-y)$ if he himself chooses Bison. If y is such that $4y = 3(1-y)$, or $3 = 7y$, or $y = 3/7$, then Fred gets the same payoff whether he chooses Stag or Bison, and also if he chooses to mix between the two in any proportions at all. But suppose Fred's mixture of Stag and Bison is such that Barney is indifferent between his pure strategies. (This game is very symmetric, so you can guess, and also calculate, that this means Fred choosing Stag a fraction $x = 4/7$ of the time.) Then Barney could be mixing in just the right proportions to keep Fred indifferent, and therefore willing to choose just the right mixture of his own. The two mixtures $x = 4/7$ and $y = 3/7$ constitute a Nash equilibrium in mixed strategies.

Is such an equilibrium satisfactory in any way? No. The problem is that the two are making these choices independently. Therefore Fred will choose Stag when Barney is choosing Bison $(4/7) \times (4/7) = 16/49$ of the time, and the other way around $(3/7) \times (3/7) = 9/49$ of the time. Thus in $25/49$ or just over half of the times the two will find themselves in separate places and get zero payoffs. Using the formulas in our calculation, we see that each gets the payoff $4 \times (3/7) + 0 \times (4/7) = 12/7 = 1.71$, which is less than the 3 of the unfavorable pure strategy equilibrium.

To avoid such errors, what they need is coordinated mixing. Can they do this while they are in their separate caves with no immediate means of communication? Perhaps they can make an agreement in advance based on something they know they are both going to observe as they set out. Suppose in their area there is a morning shower on half of the days. They can make an agreement that they both will go stag hunting if it is raining and bison hunting if it is dry. Then each will get an average payoff of $1/2 \times 3 + 1/2 \times 4 = 3.5$. Thus coordinated randomization provides them with a neat way to split the difference

between the favored and unfavored pure strategy Nash equilibria, that is, as a negotiation device.

The uncoordinated Nash equilibrium in mixed strategies not only has low payoff, but it is also fragile or unstable. If Fred's estimate of Barney's choosing Stag tips ever so slightly above $3/7 = 0.42857$, say to 0.43, then Fred's payoff from his own Stag, namely $4 \times 0.43 + 0 \times 0.57 = 1.72$, exceeds that from his own Bison, namely $0 \times 0.43 + 3 \times 0.57 = 1.71$. Therefore Fred no longer mixes but chooses pure Stag instead. Then Barney's best response is also pure Stag, and the mixed strategy equilibrium breaks down.

Finally, the mixed strategy equilibrium has a strange and unintuitive feature. Suppose we change Barney's payoffs to 6 and 7 instead of 3 and 4, respectively, leaving Fred's payoff numbers unchanged. What does that do to the mixture proportions? Again write y for the fraction of the time Barney is thought to choose Stag. Then Fred still gets $4y$ from his own choice of pure Stag and $3(1-y)$ from his own choice of pure Bison, leading to $y = 3/7$ to keep Fred indifferent and therefore willing to mix. However, writing x for the proportion of Stag in Fred's mixture, Barney gets $6x + 0(1-x) = 6x$ from his own pure Stag and $0x + 7(1-x) = 7(1-x)$ from his own pure Bison. Equating the two, we have $x = 7/13$. Thus the change in Barney's payoffs leaves his own equilibrium mixture unaffected, but changes Fred's equilibrium mixture proportions!

On further reflection, this is not so strange. Barney may be willing to mix only because he is unsure about what Fred is doing. So the calculation involves Barney's payoffs and Fred's choice probabilities. If we set the resulting expressions equal and solve, we see that Fred's mixture probabilities are "determined by" Barney's payoffs. And vice versa.

However, this reasoning is so subtle, and at first sight so strange, that most players in experimental situations fail to figure it out even when prompted to randomize. They change their mixture probabilities when their own payoffs change, not when the other player's payoffs change.

MIXING IN BUSINESS AND OTHER WARS

Our examples of the use of mixed strategies came from the sporting world. Why are there so few instances of randomized behavior out in the "real" worlds of business, politics, or war? First, most of those games are non-zero-sum, and we saw that the role of mixing in those situations is more limited and fragile, and not necessarily conducive to good outcomes. But other reasons also exist.

It may be difficult to build in the idea of leaving the outcome to chance in a

corporate culture that wants to maintain control over the outcome. This is especially true when things go wrong, as they must occasionally when moves are chosen randomly. While (some) people understand that a football coach has to fake a punt once in a while in order to keep the defense honest, a similarly risky strategy in business can get you fired if it fails. But the point isn't that the risky strategy will always work, but rather that it avoids the danger of set patterns and predictability.

One application in which mixed strategies improve business performance is price discount coupons. Companies use these to build market share. The idea is to attract new customers without giving the same discount to your existing market. If competitors simultaneously offer coupons, then customers don't have any special incentive to switch brands. Instead, they stay with their current brand and take the discount. Only when one company offers coupons while the others don't are new customers attracted to try the product.

The price coupon game for competitors such as Coke and Pepsi is then analogous to the coordination problem of the hunters. Each company wants to be the only one to give coupons, just as Fred and Barney each want to choose his own favored hunting ground. But if they try to do this simultaneously, the effects cancel out and both are worse off. One solution would be to follow a predictable pattern of offering coupons every six months, and the competitors could learn to alternate. The problem with this approach is that when Coke predicts Pepsi is just about to offer coupons, Coke should step in first to preempt. The only way to avoid preemption is to keep the element of surprise that comes from using a randomized strategy.

Of course, independent randomization runs the risk of "mistakes" exactly as in our story of the Stone Age hunters Fred and Barney. Competitors can do much better by cooperating instead, and there is strong statistical evidence that Coke and Pepsi reached just such a cooperative solution. There was a span of 52 weeks in which Coke and Pepsi each offered price promotions for 26 weeks, without any overlap. If each was choosing to run a promotion in any one week at random with a 50 percent chance, and choosing this independently of the other, the chance of there being zero overlaps is $1/495918532948104$, or less than 1 in a quadrillion (a billion billion)! This was such a startling finding that it made its way to the media, including the CBS program *60 Minutes*.⁹

The purpose of the coupons is to expand market share. But each firm realizes that to be successful, it has to offer promotions when the other is not offering similar promotions. The strategy of randomly choosing weeks for promotion offers may have the intention of catching the other off-guard. But when both firms are following similar strategies, there are many weeks when both offer

promotions. In those weeks their campaigns will merely cancel each other out; neither firm increases its share and both make a lower profit. The strategies thus create a prisoners' dilemma. The firms, being in an ongoing relationship, recognize that both can do better by resolving the dilemma. The way to do this is for each company to take a turn at having the lowest price, and then once the promotion ends, everyone goes back to their regular brands. That is just what they did.

There are other cases in which businesses must avoid set patterns and predictability. Some airlines offer discount tickets to travelers who are willing to buy tickets at the last minute. But they won't tell you how many seats are left in order to help you estimate the chances of success. If last-minute ticket availability were more predictable, then there would be a much greater possibility of exploiting the system, and the airlines would lose more of their otherwise regular paying customers.

The most widespread use of randomized strategies in business is to motivate compliance at a lower monitoring cost. This applies to everything from tax audits to drug testing to parking meters. It also explains why the punishment should not necessarily fit the crime.

The typical fine for illegal parking at a meter is many times the meter fee. If the meter rate is a dollar per hour, would a fine of \$1.01 suffice to keep people honest? It would, provided the traffic police were *sure* to catch you each time you parked without putting money in the meter. Such enforcement would be very costly. The salaries of the traffic wardens would be the largest item, but the cost of administering the collection mechanism needed to keep the policy credible would be quite substantial, too.

Instead, the authorities use an equally effective and less costly strategy, namely to have larger fines and relax the enforcement efforts. When the fine is \$25, a 1 in 25 risk of being caught is enough to keep you honest. A much smaller police force will do the job, and the fines collected will come closer to covering the administrative costs.

This is another instance of the usefulness of mixed strategies. It is similar to the soccer example in some ways, and different in other respects. Once again, the authorities choose a random strategy because it is better than any systematic action: no enforcement at all would mean misuse of scarce parking places, and a 100 percent enforcement would be too costly. However, the other side, the parking public, does not necessarily have a random strategy. In fact the authorities want to make the enforcement probability and the fine large enough to induce the public to comply with the parking regulations.

Random drug testing has many of the same features as parking meter

enforcement. It is too time-consuming and costly to test every employee every day for evidence of drug use. It is also unnecessary. Random testing will uncover those who are unable to work drug free and discourage others from recreational use. Again, the probability of detection is small, but the fine when caught is high. This is one of the problems with the IRS audit strategy. The penalties are small given the chances of getting caught. When enforcement is random, it must be that the punishment is worse than the crime. The rule should be that the *expected* punishment should fit the crime, where expectation is in the statistical sense, taking into account the chance of being caught.

Those hoping to defeat enforcement can also use random strategies to their benefit. They can hide the true crime in the midst of many false alarms or decoys, and the enforcer's resources become spread too thin to be effective. For example, an air defense must be able to destroy nearly 100 percent of all incoming missiles. A cost-effective way of defeating the air defense is for the attacker to surround the real missile with a bodyguard of decoys. It is much cheaper to build a decoy missile than the real thing. Unless the defender can perfectly distinguish among them, he will be required to stop all incoming missiles, real and fake.

The practice of shooting dud shells began in World War II, not by the intentional design of building decoy missiles, but as a response to the problem of quality control. "The elimination of defective shells in production is expensive. Someone got the idea then of manufacturing duds and shooting them on a random basis. A military commander cannot afford to have a delayed time bomb buried under his position, and he never knew which was which. The bluff made him work at every unexploded shell that came over."¹⁰

When the cost of defense is proportional to the number of missiles that must be shot down, attackers can make this enforcement cost unbearably high. This problem is one of the major challenges in the design of the Star Wars defense; it may have no solution.

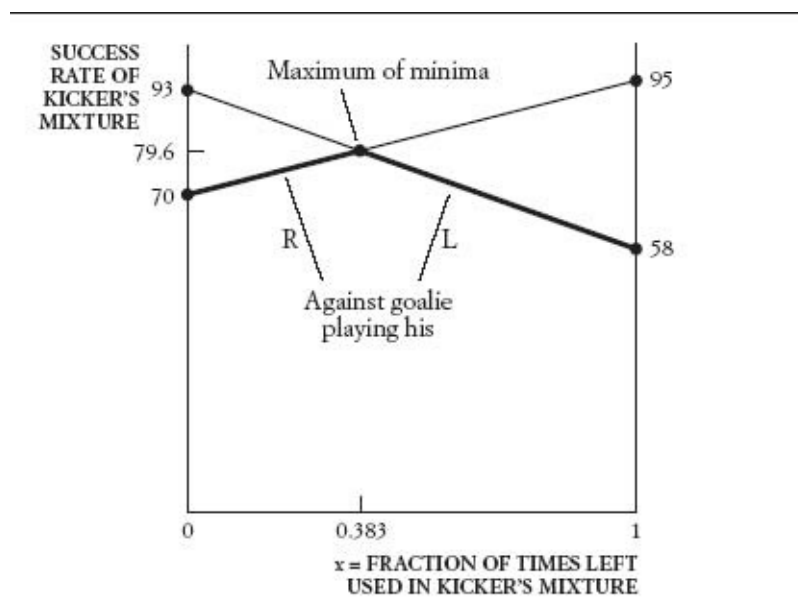
HOW TO FIND MIXED STRATEGY EQUILIBRIA

Many readers will be content to understand mixed strategies at a qualitative conceptual level and leave the calculation of the actual numbers to a computer program, which can handle mixed strategies when each player has any number of pure strategies, some of which may not even be used in the equilibrium.¹¹ These readers can skip the rest of this chapter without any loss of continuity. But for those readers who know a little high-school algebra and geometry and want

to know more about the method of calculation, we provide a few details.¹²

First consider the algebraic method. The proportion of Left in the kicker's mixture is the unknown we want to solve for; call it x . This is a fraction, so the proportion of Right is $(1-x)$. The success rate of the mixture against the goalie's Left is $58x + 93(1-x) = 93 - 35x$ percent, and that against the goalie's Right is $95x + 70(1-x) = 70 + 25x$. For these two to be equal, $93 - 35x = 70 + 25x$, or $23 = 60x$, or $x = 23/60 = 0.383$.

We can also find the solution graphically by showing the consequences of various mixes in a chart. The fraction of times Left figures in the kicker's mixture, which we have labeled x , goes horizontally from 0 to 1. For each of these mixtures, one of the two lines shows the kicker's success rate when the goalie chooses his pure strategy Left (L), and the other shows the kicker's success rate when the goalie chooses his pure strategy Right (R). The former line starts at the height 93, namely the value of the expression $93 - 35x$ when x equals zero, and descends to 58, the value of the same expression when x equals 1. The latter line starts at the vertical position of 70, namely the value of the expression $70 + 25x$ when x equals zero, and rises to 95, the value of the same expression when x equals 1.



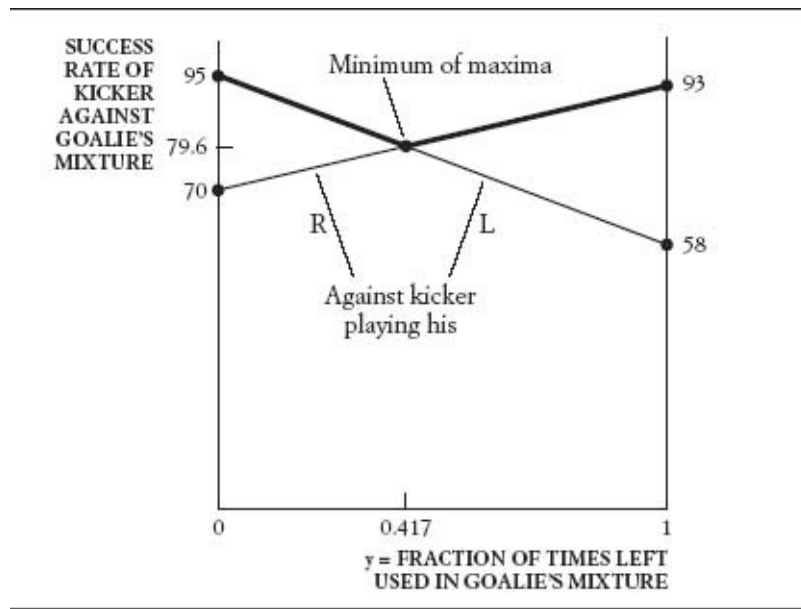
The goalie wants to keep the kicker's success rate as low as possible. Therefore if the composition of the kicker's mixture were revealed to the goalie, he would choose L or R, whichever gives the lower of the two lines. These portions of the two lines are shown thicker, forming an inverted V of the minimum success rates the kicker can expect when the goalie exploits the

kicker's choice optimally for his own purpose. The kicker wants to choose the highest success rate among these minima. He does this at the apex of the inverted V, where the two lines intersect. Close inspection, or algebraic solution, shows this to be where $x = 0.383$, and the success rate is 79.6 percent.

We can similarly analyze the goalie's mixing. Let y denote the fraction of times Left figures in the goalie's mixture. Then $(1-y)$ is the fraction of times the goalie uses his Right. If the kicker plays his L against this mixture, his average success rate is $58y + 95(1-y) = 95 - 37y$. If the kicker plays his R against this mixture, his average success rate is $93y + 70(1-y) = 70 + 23y$. For the two expressions to be equal, $95 - 37y = 70 + 23y$, or $25 = 60y$, or $y = 25/60 = 0.417$.

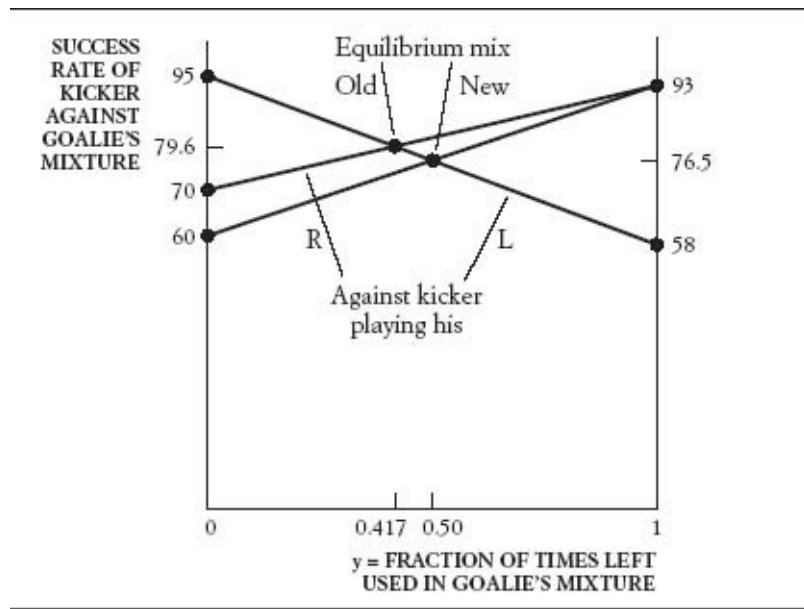
The graphical analysis from the goalie's perspective is a simple modification of that for the kicker. We show the consequences of various mixtures chosen by the goalie graphically. The fraction y of times the goalie's Left is included in his mixture goes horizontally from 0 to 1. The two lines show the kicker's success rate against these mixtures, one corresponding to the kicker's choice of his L and the other corresponding to the kicker's choice of his R. For any mixture chosen by the goalie, the kicker does best by choosing L or R, whichever gives him the higher success rate. The thicker portions of the lines show these maxima as a V shape. The goalie wants to keep the kicker's success rate as low as possible. He does so by setting y at the bottom of the V—that is, by choosing the minimum of the maxima. This occurs at $y = 0.417$, and the kicker's success rate is 79.6 percent.

The equality of the kicker's maximum of minima (maximin) and the goalie's minimum of maxima (minimax) is just von Neumann and Morgenstern's minimax theorem in action. Perhaps more accurately it should be called the "maximin-equals-minimax theorem," but the common name is shorter and easier to remember.



Surprising Changes in Mixtures

Even within the domain of zero-sum games, mixed strategy equilibria have some seemingly strange properties. Return to the soccer penalty kick and suppose the goalie improves his skill at saving penalties struck to the natural (Right) side, so the kicker's success rate there goes down from 70 percent to 60 percent. What does this do to the goalie's mixture probabilities? We get the answer by shifting the relevant line in the graph. We see that the goalie's use of Left in his equilibrium mix goes up from 41.7 percent to 50 percent. When the goalie improves his skill at saving penalties struck to the right, he uses that side less frequently!



Although this seems strange at first sight, the reason is easy to understand. When the goalie gets better at saving penalties struck to the right, the kicker will kick to the right less frequently. Responding to the fact that more shots are being struck to the left, the goalie chooses that side in greater proportion in his mixture. The point of improving your weakness is that you don't have to use it so often.

You can verify this by recalculating the kicker's mixture in response to this change; you will see that the proportion of Left in his mixture goes up from 38.3 percent to 47.1 percent.

And the goalie's work on his right-side skill does yield a benefit: the average rate of goal scoring in the equilibrium goes down from 79.6 percent to 76.5 percent.

Upon reflection, the seeming paradox has a very natural gametheoretic logic after all. What is best for you depends not only on what you do but what other players do. That is what strategic interdependence is, and should be, all about.

CASE STUDY: JANKEN STEP GAME*

The scene is a sushi bar in downtown Tokyo. Takashi and Yuichi are sitting at the bar drinking sake while waiting for their sushi. Each has ordered the house specialty, uni sashimi (sea urchin). Unfortunately, the chef reports that he has only one serving of uni left. Who will defer to the other?

In America, the two might flip a coin. In Japan, the two would more likely

play the Janken game, better known in the West as Rock Paper Scissors. Of course, by now you are experts in RPS, so to make the problem a little more challenging, we introduce a variant called the Janken step game.

The Janken step game is played on a staircase. As before, the two players simultaneously cast rock, paper, or scissors. But now, the winner climbs up the staircase: five steps if the winner played paper (five fingers); two steps for winning with scissors (two fingers); one step for winning with rock (no fingers). Ties are replayed. Normally, the winner is the first to the top of the stairs. We simplify the game slightly by assuming that each player's goal is just to get as far ahead of the other player as possible.

What is the equilibrium mixture of strategies for this version of the Janken step game?

Case Discussion

Since each step puts the winner further ahead and the loser that much further behind, we have a zero-sum game. Considering all possible pairs of moves leads to the following game table. The payoffs are measured in terms of steps ahead.

		Yuichi's choice		
		Rock	Paper	Scissors
Takashi's choice	Rock	0	5	-1
	Paper	-5	0	2
	Scissors	1	-2	0
		0	-5	2

How can we find the equilibrium mixture of throwing Paper, Scissors, and Rock? Earlier we showed you some simple numerical calculations and graphical methods that are useful when each side has only two alternatives, like forehand and backhand. But in the Janken step game, there are three alternatives.

The first question to ask is what strategies will be part of the equilibrium mixture. Here the answer is that all three are essential. To confirm this, imagine that Yuichi never plays Rock. Then Takashi would never play Paper, in which case Yuichi would never use Scissors. Continuing along this line implies that Takashi would never use Rock, thus Yuichi would never use Paper. The

assumption that Yuichi never uses Rock eliminates all of his strategies and so, must be false. A similar argument demonstrates that the other two strategies are indispensable to Yuichi's (and Takashi's) mixing equilibrium.

We now know that all three strategies must be used in the equilibrium mixture. The question becomes when will all three strategies be used. The players are interested in maximizing their payoffs, not mixing for mixing's sake. Yuichi is willing to randomize between Rock, Paper, and Scissors if and only if all three options are equally attractive. (If Rock offered Yuichi a higher payoff than either Paper or Scissors, then he should play Rock exclusively; but that would not be an equilibrium.) Thus, the special case when all three strategies give Yuichi the same expected payoff is what defines Takashi's equilibrium mixture.

Let us suppose that Takashi uses the following mixing rule:

p = probability that Takashi casts paper;
 q = probability that Takashi casts scissors;
 $1-(p + q)$ = probability that Takashi casts rock.

Then if Yuichi plays rock, he will fall behind five steps if Takashi plays paper (p) and win one step if Takashi plays scissors (q), for a net payoff of $-5p + q$. In the same way, Yuichi would get the following payoffs from each of his strategies:

Rock: $-5p + 1q + 0(1-(p + q)) = -5p + q$
 Scissors: $2p + 0q - 1(1-(p + q)) = 3p + q - 1$
 Paper: $0p - 2q + 5(1-(p + q)) = -5p - 7q + 5$

Yuichi will find the three options equally attractive only when

$$-5p + q = 3p + q - 1 = -5p - 7q + 5$$

Solving these equations reveals: $p = 1/8$, $q = 5/8$, and $(1-p-q) = 2/8$.

This defines Takashi's equilibrium mixture. The game is symmetric, so Yuichi will randomize according to the same probabilities.

Note that when both Yuichi and Takashi use their equilibrium mixture, their expected payoff from each strategy is zero. While this is not a general feature of

mixed strategy outcomes, it is always true for symmetric zero-sum games. There is no reason why Yuichi should be favored over Takashi, or vice versa.

In chapter 14, “Fooling All the People Some of the Time: The Las Vegas Slots” offers another case study on choice and chance.